Why Riding the Lightning? Equilibrium Analysis for Payment Hub Pricing

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Abstract—**Payment Channel Network (PCN) is an auspicious solution to the scalability issue of the blockchain, improving transaction throughput without relying on on-chain transactions. In a PCN, nodes can set prices for forwarding payments on behalf of other nodes, which motivates participation and improves network stability. Analyzing the price setting behaviors of PCN nodes plays a key role in understanding the economic properties of PCNs, but has been under-studied in the literature. In this paper, we apply equilibrium analysis to the price-setting game between two payment hubs in the PCN with limited channel capacities and partial overlap demand. We analyze existence of pure Nash Equilibriums (NEs) and bounds on the equilibrium revenue under various cases, and propose an algorithm to find all pure NEs. Using real data, we show bounds on the price of anarchy/stability and average transaction fee under realistic network conditions, and draw conclusions on the economic advantage of the PCN for making payment transfers by cryptocurrency users.**

Index Terms—**Payment channel network, game theory, price-setting competition, Bertrand competition**

I. INTRODUCTION

Blockchain-based cryptocurrencies such as the Bitcoin [1] can execute transactions without a trust-based model, guaranteeing security via decentralized consensus [2]. But compared to existing payment systems, blockchains have poor throughput and high settlement latency [3]. The Payment Channel Network (PCN) has emerged as a well-known solution to the blockchain scalability issue. When two nodes open a payment channel, an infinite number of transactions can be settled between the two nodes without involving the blockchain, as long as the channel capacity is not exceeded. A PCN is a network of nodes interconnected by payment channels, where Hash Time Locked Contracts (HTLCs) [4] can be used to construct a payment path between non-adjacent nodes. Each intermediate node on the path (called a *router*) charges a small fee for forwarding the payment, to compensate for its cost. Using off-chain channels for payments can greatly improve throughput and settlement speed. Moreover, current PCNs have much lower transaction fees than the blockchain, incentivizing cryptocurrency users to use PCNs for their payments. As an example of a PCN, the Lightning Network (LN) of Bitcoin, first deployed in 2017, already has 17,671 nodes and 79,557 open channels as of 2021, and has a total network capacity of 3,215 Bitcoins [5].

In a PCN, a user's goal is to make payments with the lowest cost, while each router sets fees to earn revenue from serving user demand. Current PCNs such as the LN do not provide fee-setting guidelines for routers to maximize revenue, and

most LN nodes set fees based on default values. Alternative fee policies were proposed to keep channels balanced [6] or maximize revenue [7]. Yet an open question remains: *if routers can set fees freely, will this lead to selfish fee setting that will increase PCN fees to be comparable to on-chain transaction fees, thus canceling the PCN's economic advantage?*

In this paper, we perform an equilibrium analysis to answer the above question. As current PCNs are highly centralized [8] with dedicated payment hubs [9], we focus on a *two-hub model*, where two independent payment hubs (routers) both provide payment services to common recipients. Senders can choose any connected hub to send payments to a recipient. Some senders are connected to only one hub, while others may have channels with both hubs. This forms a competitive market between the two hubs, with both locked-in and shared demand. Based on senders' demand and valuations, each hub sets its prices (unit-demand transaction fees) to maximize revenue, considering its opponent's price setting. The resulting game is a generalized Bertrand competition [10] with non-continuous demand curves, locked-in demand, and capacity constraints.

We analyze existence of pure Nash Equilibriums (NEs) in various cases of the game, and develop an algorithm to find all pure NEs if any exists; when pure NEs do not exist, we derive lower and upper bounds on the equilibrium revenue. Utilizing these, we characterize the achieved revenue under competition in realistic PCN scenarios, and compare it to the optimal revenue achievable when the hubs are *cooperative* instead of *competitive*. From the result, we conclude with a preliminary answer to the above question, *i.e., the competitive nature of PCN will ultimately make its transaction fee much lower than the blockchain, especially when the network capacity becomes larger and larger.* While our analysis is based on the two-hub model, we believe our conclusion holds for the case with more routers due to the even increased competition.

Our main contributions are summarized as follows:
- We formulate the price-setting game between two payment hubs in a PCN as a generalized Bertrand game.
- We analyze existence of pure NEs for special and general cases, and characterize equilibrium revenue of a game.
- We design an efficient algorithm to find all pure NEs.
- Utilizing real-world datasets, we analyze price of anarchy and price of stability under various capacities, and draw conclusions on PCN prices compared to the blockchain.

Organization: §II presents our system model. §III formulates the game. §IV analyzes special-case pure NEs and derives NE revenue bounds. §V analyzes the general case and presents our NE finding algorithm. §VI shows simulation results. §VII reviews related work. §VIII concludes the paper.

II. System Model

We consider a PCN consisting of two non-cooperative payment hubs acting as routers, and n users acting as senders and/or recipients of payments. Each router $i \in \{1,2\}$ has payment channels with some users. A channel is divided into two links, one from the user to the router (the *uplink*), and one from the router to the user (the *downlink*). Each link has a balance value. A channel has a total capacity, *i.e.*, an upper limit on the sum of balance values on both directions.

A payment from a sender to a recipient goes through exactly two links: an uplink, and a downlink. A payment is executed via a smart contract (*e.g.*, an HTLC) across the two links, and will occupy the balance value equal to its payment amount (plus transaction fee) on each link during execution; if either link has insufficient balance, the payment is rejected. When the contract completes, the payment amount will be added to the balances of the reverse directions on both channels, while consuming the balance values on the forward directions.

Based on standard PCN protocols (*e.g.*, the LN protocol), the transaction fee of a link is set by its originating node. Hence, each router can set the fee of its downlink towards a recipient. We only consider fees proportional to payment amounts, as base fees are commonly negligible and may be removed in near future [11]. When both routers are connected to the same recipient, each router can set the fee for its own downlink to maximize its revenue over the next period, which forms the basis of the non-cooperative game between the routers. We assume routers have synchronized fee-setting periods, and each period is sufficiently short such that the downlink balances will not be recharged by completed backward transactions. This makes the fee-setting game for each recipient independent within each period. Hence we only consider one recipient in game formulation here and hereafter. Let t_i be the starting balance of router i's downlink to the recipient at the start of the next period, and let p_i be its proportional fee (price). With slight abuse of terminology, we call the starting downlink balance of router i its *capacity*. We assume each router has a *reserved price* $RE > 0$, below which the router will not forward payments due to lack of revenue. For simplicity, we assume both routers have the same RE.

Let Ω be the set of senders making payments to the recipient in the next period. Below we use the words "sender" and "user" interchangeably. We divide the senders into Ω_i, which contains *locked-in users* that are only connected to router i for $i \in \{1,2\}$; and Ω_Λ, which contains *overlap users* that are connected to both routers simultaneously. Let $n = |\Omega|$. We assume each sender has enough uplink balance to make payment to the recipient in the next period.

Each sender k has a payment demand δ_k in the next period and a cost upper bound c_k. c_k is the maximum price that it can accept for paying via the PCN; if there is no path with price no larger than c_k, the sender will instead use the blockchain. *W.l.o.g.*, we assume $c_k \geq RE$ for any user k. For user sets $\Omega_1, \Omega_2, \Omega_\Lambda$, we define sets C_1, C_2, C_Λ as the sets of cost upper bounds of the corresponding user sets respectively, and define
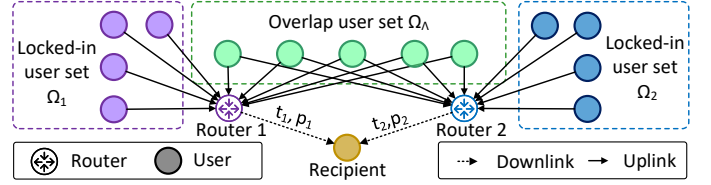

Fig. 1. System model of the price-setting game. The senders are divided into 3 sets based on the connection to the routers. The sender and recipient connected to the router through uplink and downlink respectively.

$C = C_1 \cup C_2 \cup C_\Lambda \cup \{RE\}$. We assume values in C, C_1, C_2, C_Λ are all sorted in ascending orders.

We assume all demands and cost upper bounds are known to both routers. In practice, the routers can learn and estimate the expected demand of senders based on each sender's historical demand towards each recipient, and we leave demand estimation using machine learning to our future work. Fig. 1 shows the system model of this paper.

III. Price-Setting Game Formulation

A. Strategy Space and Demand Function

The strategy space of each router is the set of feasible prices, $Y = [RE, c_{\max}]$, where $c_{\max} \in C$ is the maximum cost upper bound of senders; any price lower than RE is not acceptable to a router, while any price higher than $c_{\max}$ leads to no revenue.

Based on sender demand and cost upper bounds, we can define the demand function for each user set Ω_τ for $\tau \in \{1,2,\Lambda\}$. Define $S_\tau(p) = \{k \in \Omega_\tau | c_k \geq p\}$ as the set of senders in Ω_τ whose cost upper bounds exceed price p, the demand function of set Ω_τ is defined as $d_\tau(p) = \sum_{k \in S_\tau(p)} \delta_k$. By its definition, $d_\tau(p)$ is a left-continuous and monotonically non-increasing step function on range Y.

B. Game Setting

We assume routers and senders are rational and selfish. In particular, sender k will choose the router i to conduct its payment if $p_i < \min\{c_k, p_{\neg i}\}$ and the capacity of router i is sufficient. For competition between the two routers, we consider a pessimistic model [12], *i.e.*, when $p_i < p_{\neg i}$, router i will choose to maximize its own revenue, while also minimizing the revenue of router $\neg i$ by serving users with the highest cost upper bounds within the overlap user set Ω_Λ before starting to serve low-value overlap users or its own locked-in users. For the special case when the prices of the two routers are equal, we assume the demand is allocated proportionally based on the remaining capacity of each router, as users are likely to randomly choose a router if both have the same price and sufficient capacity.

C. Utility Function

Based on the demand function and game setting above, the utility (revenue) function of router i is $\Pi_i(p_i, p_{\neg i}) =$

$$\begin{cases} L_i(p_i) \equiv p_i \cdot \min\{t_i, d_i(p_i) + d_\Lambda(p_i)\}, & \text{if } p_i < p_{\neg i}, \\ \Phi_i(p) \equiv p \cdot \min\{t_i, d_i(p) + \phi_i d_\Lambda(p)\}, & \text{if } p_i = p_{\neg i} = p, \quad (1) \\ M_i(p_i) \equiv p_i \cdot \min\{t_i, d_i(p_i) + \psi_i\}, & \text{if } p_i > p_{\neg i}, \end{cases}$$

where $\phi_i = \frac{\max\{0, t_i - d_i(p)\}}{\max\{0, t_i - d_i(p)\} + t_{\neg i}}$, $\psi_i = \max\{0, d_\Lambda(p_i) - t_{\neg i}\}$. This utility function includes three parts. $L_i(p_i)$ is the revenue

when router i's price is less than router $\neg i$. The overlap users will choose router i first, and the revenue of router i is p_i multiplied by the minimum of the capacity of router i and the total demand of Ω_i and Ω_Λ users. $\Phi_i(p)$ is the revenue when the prices of two routers are equal, the overlap demand is distributed proportionally to the remaining capacity as mentioned before. $M_i(p_i)$ is the revenue when router i's price is greater than router $\neg i$. The demand $d_i(p_i)$ of locked-in users is still available, but only remaining of the overlap demand after being served by router $\neg i$ is available to router i, which is $\max\{0, d_\Lambda(p_i) - t_{\neg i}\}$. In any case, we call the actual demand $\Pi_i(p_i, p_{\neg i})/p_i$ served by router i its *effective demand*.

D. Nash Equilibrium Definitions

Here we outline definitions used in our equilibrium analysis.

Definition 1. *Router i's set of **best responses** $BR_i(p_{\neg i})$ is the set of strategies that maximize router i's revenue in response to the price $p_{\neg i}$ set by router $\neg i$:*

$$BR_i(p_{\neg i}) = \arg\max_{p \in Y}\{\Pi_i(p, p_{\neg i})\}, \quad \forall i \in \{1, 2\}. \quad (2)$$

Definition 2. *A **pure Nash Equilibrium (NE)** of the above price-setting game is a strategy profile $(p_1^*, p_2^*) \in Y \times Y$, such that for $\forall i \in \{1, 2\}$, p_i^* is a best response: $p_i^* \in BR_i(p_{\neg i}^*)$. In other words, no router i can unilaterally change its price to an alternative pure strategy $p_i' \in Y$ and get a higher payoff:*

$$\Pi_i(p_i^*, p_{\neg i}^*) \geq \Pi_i(p_i', p_{\neg i}^*), \quad \forall i \in \{1, 2\}. \quad (3)$$

Definition 3. *A **mixed strategy** of router i is a probability distribution over Y, $\sigma_i : Y \to \mathbb{R}^*$, such that $\sum_{p \in Y} \sigma_i(p) = 1$. Let ξ be the set of possible mixed strategies, a **mixed-strategy NE** is a (mixed) strategy profile $(\sigma_1^*, \sigma_2^*) \in \xi \times \xi$ such that:*

$$\Pi_i(\sigma_i^*, \sigma_{\neg i}^*) \geq \Pi_i(\sigma_i, \sigma_{\neg i}^*), \quad \forall \sigma_i \in \xi, i \in \{1, 2\}. \quad (4)$$

In our analysis, we will mainly focus on the pure NEs in a game. When pure NEs are non-existent, we will instead try to bound the total equilibrium revenue under mixed-strategy NEs, without deriving them directly. We leave detailed analysis of mixed-strategy NEs to our future work.

IV. PRICE-SETTING GAME ANALYSIS

We start off with analysis of known pure NEs in special cases, and then derive lower and upper bounds on the equilibrium revenue of a general-case game where pure NE may not exist.

A. Monopoly Price and Monopoly Revenue

We first define the *monopoly price* given a set of senders.

Definition 4. *Given a set of senders $\Omega_* \subseteq \Omega$ for router i, its **monopoly price** is defined as $\tilde{p}_i(\Omega_*) \in \arg\max_{p \in Y}\{p \cdot \min\{t_i, d_*(p)\}\}$, where $d_*(p)$ is the demand curve of Ω_*. The **monopoly revenue** is $\Upsilon_i(\Omega_*) = \max_{p \in Y}\{p \cdot \min\{t_i, d_*(p)\}\}$.*

Next we show that $\Upsilon_i(\Omega_*)$ only contains at prices in C.

Lemma 1. *For $\forall i$, $\Upsilon_i(\Omega_*) = \max_{p \in C}\{p \cdot \min\{t_i, d_*(p)\}\}$.*

Proof. Note that prices in C are sorted in ascending order. The set $S_*(p)$ and hence $d_*(p)$ for Ω_* does not change when $p \in (c_{k-1}, c_k]$ where $c_{k-1}, c_k \in C$. Since $d_*(p) > 0$ and $p > RE > 0$, $\Upsilon_i(\Omega_*)$ is monotonically non-decreasing when $p \in (c_{k-1}, c_k]$, at least one p maximizing $p \cdot \min\{t_i, d_*(p)\}$ appears at the right end point $c_k \in C$ of a section $(c_{k-1}, c_k]$. $\square$

Lemma 2. *For router i, its monopoly revenue $\Upsilon_i(\Omega_*)$ for any sender set Ω_* can be computed in polynomial time.*

Proof. Based on Lemma 1, router i can iterate over C to find the maximum revenue $\Upsilon_i(\Omega_*)$ in polynomial time. $\square$

B. Special Case Analysis and Implication for General Case

Next we characterize existence of pure NEs in special cases.

1) With locked-in demand only ($\Omega_\Lambda = \emptyset$): When there is no overlap demand, each router can set a monopoly price to obtain maximum revenue from its locked-in users. Each pure NE in this case consists of their monopoly prices: $(\tilde{p}_1(\Omega_1), \tilde{p}_2(\Omega_2))$.

2) With overlap demand only ($\Omega_1 = \Omega_2 = \emptyset$):

a) Sufficient capacity: If both routers have efficient capacity, the game turns into a Bertrand competition [10]. The following theorem shows the existence of unique pure NE.

Theorem 1. *(**Bertrand duopoly equilibrium [10]**) The only NE in the Bertrand model occurs at (RE, RE).*

Under the Bertrand model, the routers will not take different prices since the higher-priced router will get 0 revenue and is always incentivized to reduce its price. If the two routers take the same price larger than RE, then either router can reduce its price by an arbitrarily small amount, and win the demand of all senders which almost doubles its revenue. Therefore, the only NE exists when both routers take RE. This shows the case with the strongest competition between the two routers.

b) Insufficient capacity: If at least one router does not have enough capacity to serve all overlap users, a variant of the Bertrand model in [12] applies (where the demand function is assumed to be continuous and decreasing). Let $t = t_1 + t_2$.

Theorem 2. *(**NE in a capacitated duopoly market [12]**) Assume $t_1 < d_\Lambda$ or $t_2 < d_\Lambda$. If function $M_i(p_i)$ uniquely reaches its maximum when both participants set the price at $P(t)$, then $(P(t), P(t))$ is a pure NE. Otherwise no pure NE exists.*

Here $P(t)$ is the price when demand is exactly equal to t. Since our demand function is left-continuous and non-increasing, we redefine $P(x)$ as the *maximum price when the demand is no less than x* as shown below:

$$P(x) = \max\{p \in Y : d_\Lambda \geq x\}. \quad (5)$$

Where d_Λ is the demand when the price equal to RE. The above theorem applies with this modified $P(\cdot)$ function. We omit the proof due to page limit.

3) The general case ($\Omega_1 \cup \Omega_2 \neq \emptyset, \Omega_\Lambda \neq \emptyset$): In the most general case where both users may have both locked-in and overlap users, pure NEs may or may not exist.

When both routers set a price reach their maximum revenue at different cases in the utility function (1), then this price pair forms a pure NE. For example, when router i and router $\neg i$ reach the maximum revenue at function L_i and $M_{\neg i}$ respectively, the corresponding strategy profile satisfies $p_i < p_{\neg i}$, so the best response of both routers can be achieved. When both routers achieve maximum at either L_i ($L_{\neg i}$) or M_i ($M_{\neg i}$) at the same time, they may compete to beat each other's price with oscillating behaviors, in which case a pure NE may not exist. The situation becomes more complicated when two router set a same price. A detailed characterization on whether a pure NE exists or not is given in the next section.

C. Upper and Lower Bounds on Equilibrium Revenue

In many cases, no pure NE exists. This could happen when best responses of both routers fall onto the range of the same function (L_i or M_i), and/or when a best response does not exist as in Eq. (6) which will be described in Sec. V. Below, we try to characterize bounds on the equilibrium revenue when mixed strategies are allowed.

Definition 5. *Equilibrium revenue R^* is the expected total revenue of any pure NE (p_1^*, p_2^*) or mixed-strategy NE (σ_1^*, σ_2^*).*

We derive bounds on R^* based on the Support Lemma [13]:

Lemma 3. *([13, Lemma 33.2]) If (σ_1^*, σ_2^*) is a mixed-strategy NE, then every pure strategy p_i in the support of σ_i^* must be a best response to $\sigma_{\neg i}^*$, for $\forall i \in \{1, 2\}$.*

A lower bound (LB) of R^* is given as follows:

Lemma 4. $R^* \geq \sum_{i=1}^{2} \max_{p_i \in Y} \{M_i(p_i)\} = R_{\mathsf{LB}}^*$.

Proof. Since $\max\{M_i(p_i)\}$ is the monopoly revenue of router i when serving only its locked-in demand and left-over of the opponent, any strategy yielding a lower revenue than R_{LB}^* cannot be a best response, cannot be in the support of σ_i^*. □

An upper bound (UB) of R^* is given as follows:

Lemma 5. $R^* \leq \max_{p_1, p_2 \in Y} \{\sum_{i=1}^{2} \Pi_i(p_i, p_{\neg i})\} = R_{\mathsf{UB}}^*$.

Proof. R_{UB}^* is an upper bound on the total revenue when any strategy profile is played, including any mixed strategy. □

In next section, we will use R_{UB}^* and R_{LB}^* to evaluate the impact of competition in cases where a pure NE does not exist. Unlike NEs, these bounds always exist in any game.

V. General-case NE Analysis and Searching

In this section, we analyze NE under general-case and propose an algorithm to discover all the pure NEs if any exists.

A. NE Analysis for the General Case

Our first result shows that best responses and thus pure NEs can only exist when both routers set prices in the set C.

Theorem 3. *Given $p_{\neg i}$ set by router $\neg i$, then every best response $p_i \in BR_i(p_{\neg i})$ satisfies that $p_i \subseteq C$.*

Proof. Let us assume $p_i \in Y \setminus C$, i.e., $p_i \in (c_{k-1}, c_k)$ for $c_{k-1}, c_k \in C$. First, when $p_i \neq p_{\neg i}$, router i uses either function $L_i(p_i)$ or $M_i(p_i)$ as its revenue function. Let $\epsilon > 0$ be an arbitrarily small amount, if the corresponding effective demand is non-zero, then router i can always increase p_i by ϵ to increase revenue. Further, router i cannot have a best response with 0 effective demand, as it can always set its price to RE and serve some demand with a positive revenue. So $p_i \in (c_{k-1}, c_k)$ cannot be a best response when $p_i \neq p_{\neg i}$.

Second, when $p_i = p_{\neg i} \in (c_{k-1}, c_k)$, then

1) If t_i is not saturated, which means $t_i > d_i(p_i) + \phi_i d_\Lambda(p_i)$, we have two cases that need to be considered and p_i cannot be a best response in both cases:
   a) $\phi_i d_\Lambda(p_i) > 0$. There is overlap demand served by router $\neg i$, hence router i reducing ϵ leads to a higher revenue $(p_i - \epsilon) \cdot \min\{t_i, d_i(p_i) + d_\Lambda(p_i)\} > p_i(d_i(p_i) + \phi_i d_\Lambda(p_i))$.
   b) $\phi_i d_\Lambda(p_i) = 0$. In this case, $t_i > d_i(p_i)$. Increasing p_i by ϵ does not change the demand, so the revenue increases.

2) If t_i is saturated (and so is $t_{\neg i}$), which means $t_i \leq d_i(p_i) + \phi_i d_\Lambda(p_i)$. First, decreasing p_i cannot help router i increase revenue since $(p_i - \epsilon)t_i < p_i t_i$ when t_i is still saturated at price $(p_i - \epsilon)$. Now, a necessary condition for p_i to be a best response is when *increasing* ϵ in price will not increase its revenue. Assuming that $p_i \in (c_{k-1}, c_k)$ is a best response, we need $M_i(p_i) < \Phi_i(p_i)$. If $M_i(p_i) = \Phi_i(p_i)$, p_i cannot be a best response since increasing by ϵ increases the revenue. The following two conditions must be satisfied:
   a) $d_i(p_i) + \psi_i < t_i$, which indicates that $d_i(p_i) < t_i$.
   b) $d_i(p_i) + \psi_i = d_i(p_i) + \max\{0, d_\Lambda(p_i) - t_{\neg i}\} < d_i(p_i) + \phi_i d_\Lambda(p_i)$. We have $d_i(p_i) + d_\Lambda(p_i) - t_{\neg i} \leq d_i(p_i) + \psi_i < d_i(p_i) + \phi_i d_\Lambda(p_i)$. Then we have $d_\Lambda(p_i) - t_{\neg i} < \phi_i d_\Lambda(p_i)$, thus $1 - \frac{t_{\neg i}}{d_\Lambda(p_i)} < 1 - \frac{t_{\neg i}}{t_i - d_i(p_i) + t_{\neg i}}$. Eventually, we need $d_\Lambda(p_i) < t_i - d_i(p_i) + t_{\neg i}$.

   As t_i is saturated, there are two possible cases: First, if $d_i(p_i) \geq t_i$, this contradicts Condition a) above. Second, if $d_i(p_i) < t_i$, we have $d_i(p_i) + \phi_i d_\Lambda(p_i) = d_i(p_i) + \frac{t_i - d_i(p_i)}{t_i - d_i(p_i) + t_{\neg i}} d_\Lambda(p_i) \geq t_i$, and hence $\frac{t_i - d_i(p_i)}{t_i - d_i(p_i) + t_{\neg i}} d_\Lambda(p_i) \geq t_i - d_i(p_i)$. Eventually we have $d_\Lambda(p_i) \geq t_i - d_i(p_i) + t_{\neg i}$. This contradicts Condition b) above. We then conclude that t_i being saturated and $M_i(p) < \Phi_i(p)$ cannot be both true.

   In summary, $p \in (c_{k-1}, c_k)$ cannot be a best response. □

Following Theorem 3, we can reduce the strategy space of each router to C, and iterate over all possible strategy profiles $(p_1, p_2) \in C \times C$ to locate all pure NEs. By the definition of pure NE in Definition 2, the necessary and sufficient condition for (p_1, p_2) to be a pure NE is when both routers' prices are within the best response sets. Hence, another necessary condition is that the best response set must be non-empty for either router given the other router's price. Given this, the following lemma is useful for eliminating non-NE price pairs:

Lemma 6. *Given $p_{\neg i}$, the best response set $BR_i(p_{\neg i}) = \emptyset$ iff*

$$\sup_{p_i \in Y}\{\Pi_i(p_i, p_{\neg i})\} = L_i(p_{\neg i}) > \Phi_i(p_{\neg i}). \quad (6)$$

Proof. Define function $\Pi_i'(p_i, p_{\neg i}) = L_i(p_i)$ when $p_i \leq p_{\neg i}$, and $\Pi_i'(p_i, p_{\neg i}) = M_i(p_i)$ when $p_i > p_{\neg i}$. Notably, $\Pi_i'(p_i, p_{\neg i})$ and $\Pi_i(p_i, p_{\neg i})$ differs on only one point when $p_i = p_{\neg i}$, in which case $\Pi_i(p_i, p_{\neg i}) = \Phi_i(p_i)$ and $\Pi_i'(p_i, p_{\neg i}) = L_i(p_i) \geq \Phi_i(p_i)$. By definition, Π_i' is left-continuous and piece-wise non-decreasing based on L_i and M_i, and hence its maximum value always exists, i.e., $\sup_{p \in Y}\{\Pi_i'(p, p_{\neg i})\} = \max_{p \in Y}\{\Pi_i'(p, p_{\neg i})\}$. If Π_i' has any maximizer at $p_i \neq p_{\neg i}$, or if Π_i' has a unique maximizer $p_i = p_{\neg i}$ but $L_i(p_i) = \Phi_i(p_i)$, the same maximizer also maximizes Π_i, in which case $BR_i(p_{\neg i}) \neq \emptyset$. If Π_i' has a unique maximizer $p_i = p_{\neg i}$ and yet $L_i(p_i) > \Phi_i(p_i)$, then there is no maximizer of Π_i, because 1) for any $p \neq p_{\neg i}$, there always exists $p' \neq p_i$ where $\Pi_i(p', p_{\neg i}) > \Pi_i(p, p_{\neg i})$, and 2) for $p = p_{\neg i}$, there exists an arbitrarily small $\epsilon > 0$ such that $\Pi_i(p - \epsilon, p_{\neg i}) = L_i(p - \epsilon) > \Phi_i(p) = \Pi_i(p, p_{\neg i})$. □

B. Pure NE Searching Algorithm

Based on Theorem 3 and Lemma 6, we propose an algorithm to find all pure NEs of this game, shown in Algorithm 1. The main idea is to first find the candidate best response sets of

two routers from the set C. Then, we add a strategy profile to the pure NE set when 1) the best response sets of both routers exist given each other's price, and 2) prices of both routers are in their best response sets respectively.

Algorithm 1: Pure NE Searching

Input: Price set C, demands d_1, d_2, d_Λ, capacities t_1, t_2.
Output: Pure NE set P.

1 Initialize an empty pure NE set P;
2 **for** $\forall p_a$ *in* C **do**
3   **for** $\forall p_b$ *in* C **do**
4   Compute $\Pi_1(p_b, p_a)$ and $\Pi_2(p_b, p_a)$ based on (1);
5     $BR'_1(p_a) \leftarrow \arg\max_{p \in C} \{\Pi_1(p, p_a)\}$;
6     $BR'_2(p_a) \leftarrow \arg\max_{p \in C} \{\Pi_2(p, p_a)\}$;
7 **for** $\forall p_1 \in C$ **do**
8   **for** $\forall p_2 \in C$ **do**
9   **if** $p_1 \in BR'_1(p_2)$ **and** $p_2 \in BR'_2(p_1)$ **then**
10   **if** $p_1 \neq p_2$ **then** $P \leftarrow P \cup \{(p_1, p_2)\}$;
11   **if** $p_1 = p_2$ **and** Eq. (6) does not hold **then**
12                 $P \leftarrow P \cup \{(p_1, p_2)\}$;

13 **return** P.

First, we initialize an empty pure NE set P. In Lines 2–6, we first compute the revenues of the two routers given any pair of prices in C, and then compute their *candidate best response sets* $BR'_i(p_{\neg i})$ respectively. In Lines 7–12, we then scan through all price pairs again, and decide if any price pair is a true NE. Specifically, if the two prices are not equal and both are within their candidate best response sets, then they are directly added to the pure NE set P. If the two prices are equal, then they have to both be check against Eq. (6). If Eq. (6) holds true for router i, then its best response set does not exist, and p_i cannot be a best response. If Eq. (6) does not hold for both routers, then (p_1, p_2) is deemed as an NE and added to P. Finally the algorithm returns the set of NEs found, which could include zero, one, or more than one NE.

Theorem 4. *Algorithm 1 outputs all pure NEs of this game, and every price pair it outputs is a pure NE.*

Proof. Combining Theorem 3 and Lemma 6, the only case when $BR'_i(p_{\neg i}) \neq BR_i(p_{\neg i})$ is when $BR'_i(p_{\neg i})$ contains a price $p_i = p_{\neg i}$ but Eq. (6) holds true and hence $BR_i(p_{\neg i}) = \emptyset$. This case is eliminated by the condition at Line 11. $\square$

Given n users, the time complexity of Algorithm 1 is $O(n^2)$.

VI. PERFORMANCE EVALUATION

A. Experiment Settings

We use the Lighting Network (LN) topology from [3]. We choose the two most connected nodes as routers 1 and 2 respectively, and all their neighbors as users. We have 390 overlap users, and 620 and 496 locked-in users for routers 1 and 2 respectively. As channel balances are hidden in the LN, we use the most frequent channel capacity value as the downlink capacity of both routers towards an artificial destination, and hence each router has $t_i = 10^6$ satoshi.
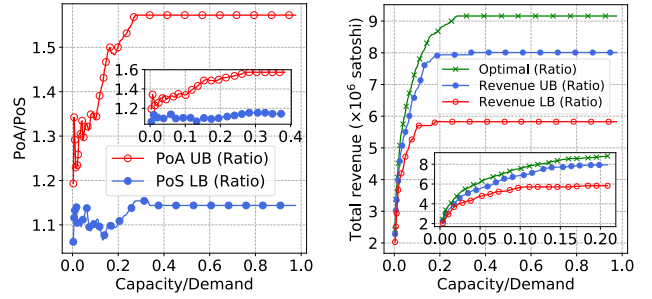
(a) PoA/PoS vs. capacity/demand   (b) Revenue vs. capacity/demand

Fig. 2. Upper and Lower Bound of PoA and/or PoS and total revenue.

We simulate user demands by randomly sampling transactions from a real-world credit card dataset [14] (translated to satoshi), and picking smallest 40% as payments since PCNs are mostly used for micropayments in reality. As paying via the blockchain incurs a fixed fee regardless of the amount, we define constant ρ as the product of each user's demand and its cost upper bound in the PCN, and let $\rho = 22367$ satoshi, the average Bitcoin transaction fee as of Oct. 28, 2021 [15]. We then generate each c_k as ρ divided by demand of the user.

To show the impact of competition, we simulate three user distributions: **Ratio**, which keeps the same user distribution as in the original LN dataset; **Overlap**, which connects all users to both routers; and **Monopoly**, where each user is only connected to one router. In addition to NE searching, we also implemented an **Optimal** algorithm, where both routers cooperate to maximize their total revenue.

We use Price of Anarchy (PoA) and Price of Stability (PoS) to measure the impact of competition. PoA (PoS) is defined as the optimal total revenue of two routers divided by the minimum (maximum) total NE revenue. We can get the PoA UB (PoS LB) using the revenue LB (UB) in Lemmas 4–5.

B. Evaluation Results

1) PoA/PoS: Fig. 2 shows the competition between two routers, with capacities of both routers increasing with the same scale. The x-axis shows the total capacity of the two routers divided by the total demand. Fig. 2(a) shows the PoS LB and PoA UB under **Ratio** distribution, which are calculated using total revenues shown in Fig. 2(b). Both bounds have increasing trends as the capacity increases. This is because the increased capacities of both routers induce more competition, and hence both routers will lower their fees to compete. When the capacity-to-demand ratio is over 0.4, both the revenues and PoA/PoS bounds become constant. This is because both routers run into full competition with enough capacity. Oscillation under 0.2 is due to randomness in both optimal and NE revenues when there is near no competition.

Fig. 3 shows the PoS LB and PoA UB under different user distributions. For **Overlap**, its PoA/PoS bounds dominate the other curves after the capacity/demand is greater than 0.4. This is because under **Overlap**, the routers run into the strongest competition where they have no revenue from locked-in users. For **Monopoly**, based on the analysis in Sec. IV-B, an NE always exists, and PoA/PoS is always 1. In this case, both routers maximize the locked-in revenues without competition.
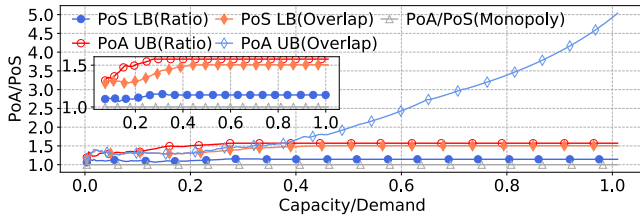
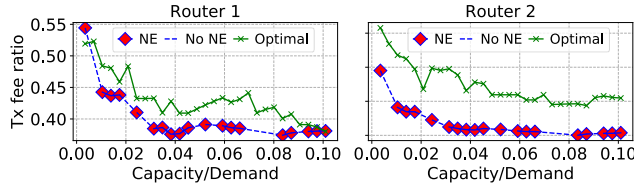Fig. 3. PoS and/or PoA bounds under different user distributions.

Fig. 4. NE and optimal transaction fees normalized by blockchain fee.

2) Transaction fee: Fig. 4 shows the ratio of average NE or optimal transaction fee over the average blockchain transaction fee under the **Ratio** distribution. The NE or optimal transaction fee is calculated *w.r.t.* all users served by the corresponding router. Dashed lines show cases when NEs do not exist. NEs exist only when capacity is low compared to the demand; when the capacity-demand ratio exceeds 0.11, no NE exists and hence we omit those points in the figure. The average NE transaction fees of both routers decrease as the capacity increases, and so do the optimal transaction fees. The NE fees are generally lower than the optimal fees, and both fees are significantly lower than the blockchain transaction fee. *To summarize, these results show that PCN has lower transaction fees than the blockchain due to router competition. The larger the network capacity, the lower the fee.*

VII. Related Work

PCN economics has only received recent attention. Engelmann *et al.* [16] studied the cheapest path routing for LN users without considering router fee setting. Paolo *et al.* [17] performed cost analysis of opening and maintaining channels in the LN. Avarikioti *et al.* [18] and Ersoy *et al.* [7] studied network creation games where nodes open channels strategically to maximize revenue. Jun Ren *et al.* [19] investigated the optimal fee structure of LN nodes from a non-competitive lens, and studied its impact on network topology. Di Stasi *et al.* [6] focused on designing the LN fee policies of routers to balance the network. None of the above addressed the price setting game between routers, or revealed the economical principles under the common belief of the low transaction fees in PCNs.

Pricing in network settings other than PCNs has been studied in the Internet literature [20]. However, as PCN differs from the Internet in that balances are not rechargeable instantly after usage, most existing solutions do not apply.

Equilibrium analysis of duopoly/oligopoly markets has been extensively studied in economics, but commonly with limiting assumptions such as linear or continuous and decreasing demand functions [12], [21] or unlimited capacities [22]. Our studied game generalizes these existing work, based on the unique properties of the PCN including locked-in users, discrete demands and limited capacities.

VIII. Conclusion

This paper focused on the economic analysis of transaction fee setting behaviors of nodes in a PCN. Equilibrium analysis was performed, including the existence of pure NEs, an algorithm for finding all pure NEs, and bounds on equilibrium revenue. With simulations using real-world datasets, we showed that PCN transaction fees can be driven down significantly by the competition between network nodes, compared to on-chain transaction fees. These preliminary results have shed light on the economical advantage of using PCNs over the blockchains, and warrant future research on PCN economics.

References

[1] S. Nakamoto, "Bitcoin: A Peer-to-Peer Electronic Cash System," 2008, accessed 2021-11-12. [Online]. Available: https://bitcoin.org/bitcoin.pdf

[2] L. Xiao, Y. Ding, D. Jiang, J. Huang, D. Wang, J. Li, and H. Vincent Poor, "A reinforcement learning and blockchain-based trust mechanism for edge networks," *IEEE Transactions on Communications*, vol. 68, no. 9, pp. 5460–5470, 2020.

[3] Y. Zhang and D. Yang, "RobustPay$^+$: Robust Payment Routing With Approximation Guarantee in Blockchain-Based Payment Channel Networks," *IEEE/ACM Transactions on Networking*, 2021.

[4] P. Joseph and T. Dryja, "The Bitcoin Lightning Network: Scalable Off-Chain Instant Payments," 2016, accessed 2021-11-12. [Online]. Available: https://lightning.network/lightning-network-paper.pdf

[5] "Bitcoin: Lightning Network Capacity," accessed 2021-11-12. [Online]. Available: https://studio.glassnode.com

[6] G. Di Stasi, S. Avallone, R. Canonico, and G. Ventre, "Routing Payments on the Lightning Network," in *Proc. IEEE Blockchain*, 2018, pp. 1161–1170.

[7] O. Ersoy, S. Roos, and Z. Erkin, "How to Profit From Payments Channels," in *Proc. FC*, 2021.

[8] S. Martinazzi and A. Flori, "The Evolving Topology of the Lightning Network: Centralization, Efficiency, Robustness, Synchronization, and Anonymity," *PloS one*, vol. 15, no. 1, p. e0225966, Jan. 2020.

[9] L. Andrea, D. D. F. Maesa, P. Mori, and L. Ricci, "Lightnings over Rose Bouquets: An Analysis of the Topology of the Bitcoin Lightning Network," in *Proc. IEEE COMPSAC*, 2021, pp. 324–331.

[10] J. Bertrand, "Book review of theorie mathematique de la richesse sociale and of recherches sur les principes mathematiques de la theorie des richesses," *Journal de Savants*, vol. 67, p. 499–508, 1883.

[11] "Zero Base Fee LN Discussion," accessed 2021-11-12. [Online]. Available: https://bitcoinops.org/en/newsletters/2021/08/25/

[12] M. J. Osborne and C. Pitchik, "Price Competition in a Capacity-Constrained Duopoly," *Journal of Economic Theory*, vol. 38, no. 2, pp. 238–260, 1986.

[13] M. J. Osborne and A. Rubinstein, *A Course in Game Theory*. MIT press, 1994.

[14] "Credit Card Fraud Detection," accessed 2021-11-12. [Online]. Available: https://www.kaggle.com/mlg-ulb/creditcardfraud

[15] "Fees Per Transaction (USD)," accessed 2021-11-14. [Online]. Available: https://www.blockchain.com/charts/fees-usd-per-transaction

[16] F. Engelmann, H. Kopp, F. Kargl, F. Glaser, and C. Weinhardt, "Towards An Economic Analysis of Routing in Payment Channel Networks," in *Proc. ACM SERIAL*, 2017, pp. 1–6.

[17] G. Paolo, G. Huberman, and C. Shikhelman, "Lightning Network Economics: Channels," *SSRN 3840374*, 2021.

[18] Z. Avarikioti, L. Heimbach, Y. Wang, and R. Wattenhofer, "Ride the Lightning: The Game Theory of Payment Channels," in *Proc. FC*, 2020, pp. 264–283.

[19] A. H. Jun Ren, L. Feng, S. A. Cheong, and R. S. Mong Goh, "Optimal Fee Structure for Efficient Lightning Networks," in *Proc. IEEE ICPADS*, Dec. 2018, pp. 980–985.

[20] T. Basar and R. Srikant, "Revenue-maximizing Pricing and Capacity Expansion in A Many-users Regime," in *Proc. IEEE INFOCOM*, 2002.

[21] R. Levitan and M. Shubik, "Price Duopoly and Capacity Constraints," *International Economic Review*, vol. 13, no. 1, pp. 111–122, 1972.

[22] C. L. Guzmán, "Price Competition on Network," *Banco de México*, 2011, accessed 2021-11-14. [Online]. Available: https://core.ac.uk/download/pdf/6289712.pdf